Comments on Some Inverted Cumulative Distributions: "Saturation in the Hausdorff Sense", Applications

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joint work with
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Introduction

Definition 1.

We consider the following two–parameters generalized inverted exponential cumulative distribution function:

$$F(t) = 1 - \left(1 - e^{-\frac{\delta}{t}}\right)^{a} \tag{1}$$

for $t > 0,\ \delta > 0,\ a > 1.$

Definition 2. (*Hausdorff (1962), Sendov (1990)*)

The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \qquad (2)$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Purposes:

• study some properties of the family (1) and prove estimate for the "saturation" - d about Hausdorff metric;

• consider modified families of adaptive functions with "polynomial variable transfer" with applications to the Antenna–feeder Analysis.

Main result

"Saturation" - d in the Hausdorff sense to the horizontal asymptote

$$F(d) = 1 - d, \tag{3}$$

i.e. d is the solution of the nonlinear equation

$$e^{\frac{\delta}{d}} - \frac{1}{1 - d^{\frac{1}{a}}} = 0.$$

Special functions

$$G(d) = e^{\frac{\delta}{d}} + K \frac{1}{d^{\frac{1}{a}}} = 0, \tag{4}$$

where

$$K(a, d) = \frac{-d^{\frac{1}{a}}}{1 - d^{\frac{1}{a}}} := K$$

$$H(d) = e^{\frac{\delta}{d} + d} - K \ln \delta^{a} = 0. \tag{5}$$

Main result

Theorem 1.

For sufficiently small values of $\delta > 0$ and $d \leq \frac{1}{8}$ for the "saturation" - d we have

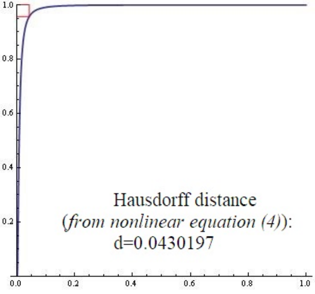
$$d \approx \frac{\delta}{\ln\ln \frac{1}{(\delta^a)^{\frac{1}{7}}}}. \tag{6}$$

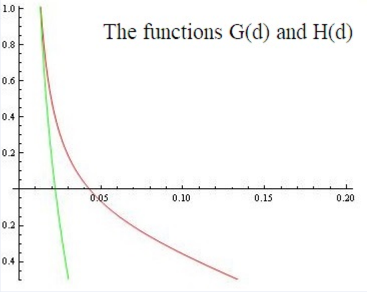
Numerical experiments

Computational examples

δ	a	d computed by (4)	d computed by (6)
0.01	2	0.0430197	0.0364409
0.001	3	0.027736	0.0147084
0.005	4	0.0123349	0.00451356
0.005	6	0.00835236	0.00330417

Case: $\delta = 0.01$; $a = 2$;





Model with "polynomial variable transfer"

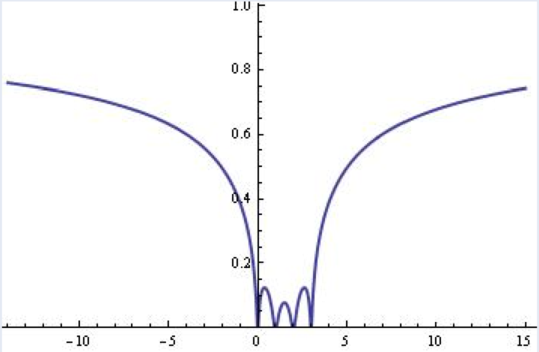
$$
\begin{aligned}
F_n^*(t) &= 1 - \left(1 - e^{-\frac{\delta}{|f(t)|}}\right)^a \\
f(t) &= \sum_{i=0}^{n} a_i t^i, \quad a_0 = 0.
\end{aligned} \tag{7}
$$

Model with "polynomial variable transfer"

Examples:

$$F_3^*(t) = 1 - \left(1 - e^{-\frac{\delta}{|t(1-t)(2-t)(3-t)|}}\right)^a .$$

A typical "filter characteristic" by using model $F_3^*(t)$ for $\delta = 2.4,\ a = 3.5$

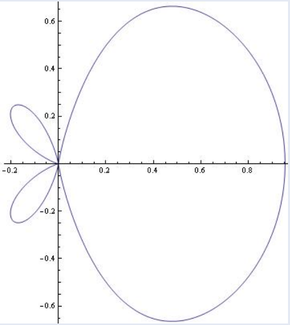


Model with "polynomial variable transfer"

Examples:

Consider the function $|F_6^*(t)|$ for $t = b\cos\theta + c$

A typical "emitting chart" using $|F_6^*(t)|$ for $n = 6,\ \delta = 0.22,\ a = 1.1,\ a_0 = 0,\ a_1 = -0.1,\ a_2 = 1.1,$ $a_3 = -1.1,\ a_4 = 0.15,\ a_5 = 0.5,\ a_6 = -0.02,\ b = -1.2,\ c = 0.001$



Some inverted cumulative distribution functions

New inverse Weibull cumulative (*Afify, Shawky, Nassar (2021)*)

$$F_1(t) = 1 - \frac{\ln\left(1 + \delta - \delta^{e^{-\frac{\delta}{t}}}\right)}{\ln \delta} \tag{8}$$

Estimate for the "saturation" - d about Hausdorff metric (*Kyurkchiev (2020)*)

$$d \approx \frac{\delta}{1 + \ln\left(\ln\left(\frac{1}{\delta}\right)\right)}, \tag{9}$$

for sufficiently small values of δ and $d \leq \frac{1}{2}$

Some inverted cumulative distribution functions

$F_2(t) = 1 - \ln\left(1 + e - e^{e^{-\frac{\delta}{t}}}\right)$	$F_3(t) = \frac{1-\left(1-e^{-\frac{\delta}{t}}\right)^a}{1+\left(1-e^{-\frac{\delta}{t}}\right)^a}$
$F_4(t) = 1 - \left(1 - e^{-\left(\frac{\delta}{t}\right)^a}\right)^b$	$F_5(t) = \frac{1-\left(1-e^{-\left(\frac{\delta}{t}\right)^a}\right)^b}{1+\left(1-e^{-\left(\frac{\delta}{t}\right)^a}\right)^b}$
$F_6(t) = \frac{e^{\alpha(1+\lambda t^{-\phi})^{-2\eta}}-1}{e^{\alpha}-1}$	$F_7(t) = \left(1 - \left(1 - e^{-\frac{\delta}{t^b}}\right)^l\right)^m$
$F_8(t) = 1 - \left(1 - \left(\frac{\alpha^{e^{-\frac{\lambda}{t}}}-1}{\alpha-1}\right)^{\phi}\right)^b$	

Model with "polynomial variable transfer"

Adaptive function of Gumbel–type $F_7(t)$

$$G_7(t) = A\left(1 - \left(1 - e^{-\delta f(t)^{-b}}\right)^l\right)^m, \tag{10}$$

where

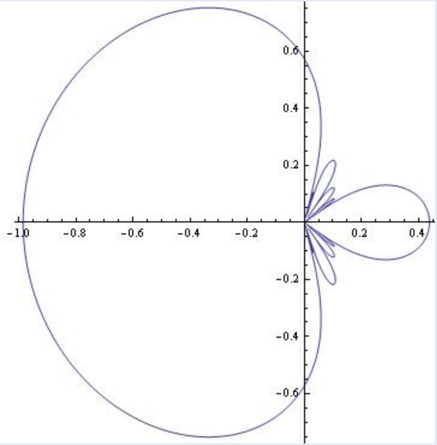
$$f(t) = \sum_{i=0}^{n} a_i t^i; \quad a_0 = 0.$$

Model with "polynomial variable transfer"

Simulation using $|G_7(\theta)|$ with

$A = 1.33$; $\delta = 2.95$; $b = 0.15$; $l = 1.5$; $m = 0.3$; $r = 1.59$; $c = -0.39$

for fixed $f(t) = t(1-t)(0.7-t)(0.5-t)$, **where** $t = r\cos\theta + c$



Remark 1

Modification of the model (8)

$$F_1(t) = 1 - \frac{\ln\left(1 + \delta - \delta^{e^{-\left(\frac{\delta}{t}\right)^{\delta}}}\right)}{\ln \delta}, \tag{11}$$

which can be considered as an adaptive function.

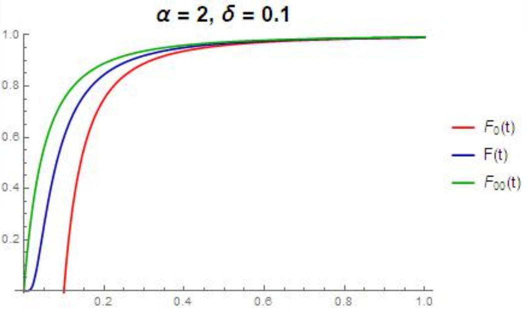
Remark 2 (technical result for the basic model $F(t)$)

Lemma 1. The following inequality holds

$$F_0(t) \leq F(t) \leq F_{00}(t),$$

where

$$F_0(t) = 1 - \left(\frac{\delta}{t}\right)^{\alpha} \quad \text{and} \quad F_{00}(t) = 1 - \left(\frac{\delta}{\delta + t}\right)^{\alpha}. \tag{12}$$



Thank you for your attention!

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